

Symplectic embeddings from concave toric domains into convex ones

Daniel Cristofaro-Gardiner*

Abstract

Embedded contact homology gives a sequence of obstructions to four-dimensional symplectic embeddings, called ECH capacities. In “Symplectic embeddings into four-dimensional concave toric domains”, the author, Choi, Frenkel, Hutchings and Ramos computed the ECH capacities of all “concave toric domains”, and showed that these give sharp obstructions in several interesting cases. We show that these obstructions are sharp for all symplectic embeddings of concave toric domains into “convex” ones. In an appendix with Choi, we prove a new formula for the ECH capacities of convex toric domains, which shows that they are determined by the ECH capacities of a corresponding collection of balls.

1 Introduction

1.1 The main theorem

It is an interesting problem to determine when one symplectic manifold embeds into another. In dimension 4, Hutchings’ “ECH capacities” give one tool for studying this question. ECH capacities are a certain sequence of nonnegative (possibly infinite) real numbers associated to any symplectic four-manifold. They are monotone under symplectic embeddings, and therefore give obstructions to symplectically embedding one symplectic 4-manifold into another.

In [3], the author, Choi, Frenkel, Hutchings, and Ramos used ECH capacities to study symplectic embeddings of “toric domains”. A *toric domain* X_Ω is the preimage of a region $\Omega \subset \mathbb{R}^2$ in the first quadrant under the map

$$\mu : \mathbb{C}^2 \rightarrow \mathbb{R}^2, \quad (z_1, z_2) \rightarrow (\pi|z_1|^2, \pi|z_2|^2).$$

Toric domains generalize ellipsoids (where Ω is a right triangle with legs on the axes) and polydisks (where Ω is a rectangle whose bottom and left

*Mathematics Department, Harvard University, Cambridge MA, USA.
Electronic address: gardiner@math.harvard.edu

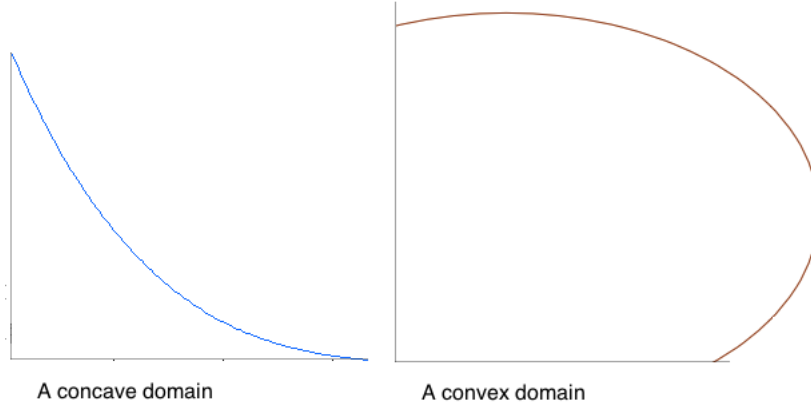


Figure 1.1: A concave toric domain and a convex one

sides are on the axes). The paper [3] computed the ECH capacities of all “concave” toric domains, and showed that these give sharp obstructions in several interesting cases, for example for all ball packings into certain unions of an ellipsoid and a cylinder.

The aim of the present article is to identify many more cases involving toric domains where ECH capacities give a sharp obstruction. It turns out that in these cases, ECH capacities can be computed purely combinatorially, and so give considerable insight into the corresponding embedding problem.

To state our main theorem, first recall the “concave toric domains” from [3]. These were defined as toric domains X_Ω , where Ω is a region in the first quadrant underneath the graph of a convex function $f : [0, a] \rightarrow [0, b]$, such that a and b are positive real numbers, $f(0) = b$, and $f(a) = 0$. We now define a related concept, see Figure 1.1.

Definition 1.1. A *convex toric domain* is a toric domain X_Ω , where Ω is a closed region in the first quadrant bounded by the axes and a convex curve from $(a, 0)$ to $(0, b)$, for a and b positive real numbers.

Note that our definition of convex toric domain differs slightly from the definition in [9].

If X is a symplectic four-manifold, let $c_k(X, \omega)$ denote the k^{th} ECH capacity of X . We can now state the main theorem of this paper:

Theorem 1.2. *Let X_{Ω_1} be a concave toric domain and let X_{Ω_2} be a convex toric domain. Then there exists a symplectic embedding*

$$\text{int}(X_{\Omega_1}) \rightarrow \text{int}(X_{\Omega_2})$$

if and only if

$$c_k(\text{int}(X_{\Omega_1})) \leq c_k(\text{int}(X_{\Omega_2}))$$

for all nonnegative integers k .

Note that an ellipsoid is both concave and convex, while a polydisc is convex. Thus, Theorem 1.2 generalizes well-known results of McDuff [13] (where X_{Ω_1} and X_{Ω_2} are both ellipsoids) and Frenkel-Müller [6] (where X_{Ω_1} is an ellipsoid and X_{Ω_2} is a polydisc). As mentioned above, a purely combinatorial formula for the ECH capacities of concave toric domains was given in [3]. In the appendix, we give a formula for the ECH capacities of convex domains that generalizes the formula from [9, Thm. 1.11], see Corollary A.5. These formulas involve counting lattice points in polygons, and the combinatorics involved can be interesting [5].

Here is an example of how one can use Theorem 1.2:

Example 1.3. Let X_{Ω_1} be an ellipsoid and let X_{Ω_2} be the convex toric domain associated to a closed symplectic toric four-manifold X . This means that Ω_2 is a Delzant polygon for X (note that any Delzant polygon is affine equivalent to a polygon Ω_2 which is convex in the sense of Definition 1.1). Then X contains the convex toric domain X_{Ω_2} , so Theorem 1.2 can be used to construct embeddings of ellipsoids into X . In fact, it is relatively straightforward to see that an ellipsoid embeds into X if and only if it embeds into X_{Ω_2} . Thus, Theorem 1.2 can be used to understand exactly when an ellipsoid embeds into a closed symplectic toric four-manifold. This is studied in [4].

More examples are given in §2.4.

We remark that it is known that ECH capacities are not always sharp, even for toric domains. A notable example of this is given by Hind and Lisi in [7], where it is shown that ECH capacities fail to be sharp for embeddings of a polydisk into a ball. Interestingly, recent work of Hutchings [9] shows that embedded contact homology can still be used to derive strong obstructions to symplectic embeddings, even when the obstructions coming from ECH capacities are weak. For example, in [9] Hutchings defines new obstructions to embedding one convex toric domain into another that can be used to recover the result of Hind and Lisi from above. It is currently not known how sharp these new obstructions are.

1.2 Idea of the proof and relationship with previous work

As mentioned above, McDuff showed that ECH capacities give a sharp obstruction to symplectically embedding one four-dimensional ellipsoid into another. Here we use a similar method. To elaborate, McDuff showed in [12] that an embedding of one rational ellipsoid into another is equivalent to a certain symplectic ball packing problem determined by the ellipsoids. In [15], it was then shown that since ECH capacities are known to be sharp for

symplectic ball packings of a ball, they are sharp for ellipsoid embeddings as well. We first show that an embedding of a “rational” concave toric domain into a rational convex one is equivalent to a symplectic ball packing problem, see Theorem 1.4, and we then use this to show that ECH capacities give a sharp obstruction to embedding a concave domain into a convex one. We remark that ball packings of a ball are relatively well-understood (indeed, they are essentially algorithmically computable [2, §2.3]), and so Theorem 1.4 is of potentially independent interest.

We now explain the details of the equivalence between embeddings of concave domains into convex ones and ball packings.

1.3 Weight sequences

In [12], McDuff introduced a sequence of real numbers determined by a 4-dimensional symplectic ellipsoid, called a *weight sequence*. Choi, the author, Frenkel, Hutchings, and Ramos generalized these weight sequences to any concave toric domain in [3]. We now review this generalization.

Let Ω be a concave toric domain. The weight sequence of Ω is a sequence of nonnegative real numbers $w(\Omega)$ defined inductively as follows. If Ω is a triangle with vertices $(0, 0)$, $(0, a)$ and $(a, 0)$, then the weight sequence of Ω is (a) . Otherwise, let $a > 0$ be the smallest real number such that Ω contains the triangle with vertices $(0, 0)$, $(0, a)$ and $(a, 0)$. Call this triangle Ω_1 . Then the line $x + y = a$ intersects the upper boundary of Ω in a line segment from $(x_1, a - x_1)$ to $(x_2, a - x_2)$, where $x_1 \leq x_2$. Let Ω'_2 be the closure of the part of Ω to the left of x_1 and above this line, and let Ω'_3 be the closure of the part of Ω to the right of x_2 and above this line, see Figure 1.2. Then, as explained in [3, §1.3], Ω'_2 is affine equivalent to a canonical concave toric domain, which we denote by Ω_2 . Similarly, Ω'_3 is affine equivalent to a canonical concave toric domain which will be denoted by Ω_3 . We now define $w(\Omega) = w(\Omega_1) \cup w(\Omega_2) \cup w(\Omega_3)$, where \cup denotes the (unordered) union with repetitions. In the inductive definition, note that $w(\Omega)$ is defined to be \emptyset if $\Omega = \emptyset$.

We now define a similar weight expansion for any convex toric domain. The definition of the weight sequence for convex toric domains is similar to the definition of the weight sequence for concave toric domains. If Ω is a triangle with vertices $(0, 0)$, $(0, b)$ and $(b, 0)$ then the weight sequence of Ω is (b) . Otherwise, let $b > 0$ be the smallest real number such that Ω is contained in the triangle with vertices $(0, 0)$, $(0, b)$ and $(b, 0)$. Call this triangle Ω_1 . The line $x + y = b$ intersects the upper boundary of Ω in a line segment from $(x_1, b - x_1)$ to $(x_2, b - x_2)$, with $x_1 \leq x_2$. Let Ω'_2 denote the closure of the portion of $\Omega_1 \setminus \Omega$ that is to the left of x_1 and below the line $x + y = b$, and let Ω'_2 denote the closure of the portion of $\Omega_1 \setminus \Omega$ that is below $b - x_2$ and below the line $x + y = b$, see Figure 1.2.

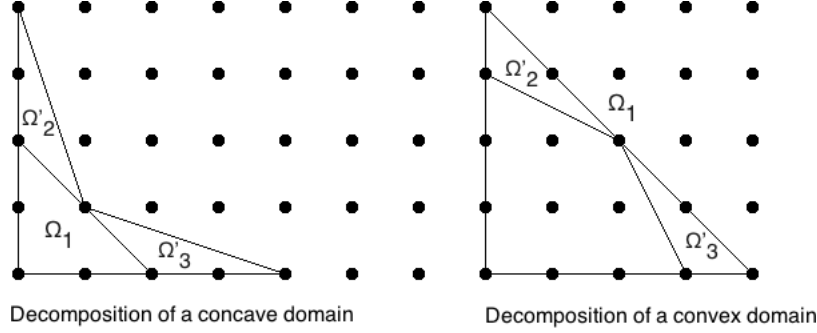


Figure 1.2: The inductive decomposition of convex and concave toric domains

The key point is now that Ω'_2 and Ω'_3 are both affine equivalent to concave toric domains, which we denote by Ω_2 and Ω_3 respectively. The equivalence for Ω'_2 is given by translating down so that the top left corner of Ω'_2 is at the origin, and then multiplying by the matrix $M = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$, while the equivalence for Ω'_3 is given by translating so that the bottom right corner is at the origin, and then multiplying by the matrix $M' = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$. We then define

$$w(\Omega) = (b; w(\Omega_2) \cup w(\Omega_3)).$$

Thus, the weight sequence for a convex toric domain consists of a number, and then an unordered set of numbers. We call the first number in this sequence the *head*, and we call the other numbers the *negative weight sequence*.

To simplify the notation, for a convex Ω , let $\widehat{B}(\Omega)$ denote the disjoint union of (closed) balls with radii given by the negative weight expansion for X_Ω . Also let $B(\Omega)$ for concave Ω denote the disjoint union of closed balls whose radii are given by the numbers in the weight expansion for Ω . Finally, call a *rational concave* domain a concave domain whose upper boundary is piecewise linear, with rational coordinates, and define a rational convex domain similarly.

We can now state the aforementioned equivalence:

Theorem 1.4. *Let X_{Ω_1} be a rational concave toric domain, let X_{Ω_2} be a rational convex toric domain, and let b be the head of the weight expansion for Ω_2 . Then there exists a symplectic embedding*

$$\text{int}(X_{\Omega_1}) \rightarrow \text{int}(X_{\Omega_2})$$

if and only if there exists a symplectic embedding

$$\text{int}(B(\Omega_1)) \sqcup \text{int}(\widehat{B}(\Omega_2)) \rightarrow \text{int}(B(b)).$$

Note that the “only if” direction of Theorem 1.4 follows from the “Traynor trick” [22], see e.g. [3, Lem. 1.8] for the version we need, and the definition of the weight expansion.

1.4 Connectivity of the space of embeddings

McDuff also showed in [12] that the space of embeddings of one ellipsoid into another is connected. To prove Theorem 1.2 and Theorem 1.4, it will be helpful to show that this also holds for embeddings of a concave domain into a convex one:

Proposition 1.5. *Let X_{Ω_1} be a concave toric domain, let X_{Ω_2} be a convex toric domain, and let g_0 and g_1 be two symplectic embeddings:*

$$X_{\Omega_1} \rightarrow \text{int}(X_{\Omega_2}).$$

Then there exists an isotopy $\Psi_t : \text{int}(X_{\Omega_1}) \rightarrow \text{int}(X_{\Omega_2})$ such that $\Psi_0 = \text{id}$ and $\Psi_1(g_1) = g_0$.

The following corollary will be particularly useful:

Corollary 1.6. *Let X_{Ω_1} be a concave domain and let X_{Ω_2} be convex. Then there is a symplectic embedding*

$$\text{int}(X_{\Omega_1}) \rightarrow \text{int}(X_{\Omega_2})$$

if and only if there is a symplectic embedding

$$X_{\lambda\Omega_1} \rightarrow \text{int}(X_{\Omega_2})$$

for all $\lambda < 1$.

1.5 ECH capacities of convex domains and ECH capacities of balls

As explained in §1.2, the fact that ECH capacities are sharp for these embedding problems essentially follows from the fact that they are sharp for symplectic ball packings of a ball. In fact, the ECH capacities of both of these domains are closely related to the ECH capacities of balls. In [3], it was shown that the ECH capacities of any concave toric domain are determined by the ECH capacities of a certain collection of balls, see [3, Thm. 1.4] for the precise statement. In an appendix with Choi, we show that this is also true for convex toric domains, see Theorem A.1.

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While this article was in its final stages, I learned that Opshtein has also recently studied symplectic embeddings of concave toric domains from a slightly different point of view, and found relationships with ball-packings, see [19, §5]. I thank Opshtein for his helpful correspondence about this.

2 Embeddings of concave toric domains into convex toric domains and ball packings

We now explain the proof of Theorem 1.4. We already showed the “only if” direction, so we now show the converse. Our proof closely follows the “inflation” method from (for example) [12, 16, 1].

2.1 Embeddings of toric domains and embeddings of spheres

The objects that we will “inflate” are certain chains of symplectic spheres. In this section, we explain the significance of these chains of spheres to our embedding problem.

The spheres that we will want to inflate arise from a sequence of symplectic blowups. We therefore start by recalling those details of the blowup construction that are relevant to us. Let L denote the homology class of the line in $\mathbb{C}P^2$. There is a symplectic form ω_0 on $\mathbb{C}P^2$, called the *Fubini-Study* form, such that $\langle \omega_0, L \rangle = 1$. Now suppose there is a symplectic embedding $\coprod_{i=1}^m B(a_i) \rightarrow (\mathbb{C}P^2, \omega_0)$. We can remove the interiors of the $B(a_i)$ and collapse their boundaries under the Reeb flow to get a symplectic manifold, called the *blowup* of the ball packing, which is diffeomorphic to $\mathbb{C}P^2 \# m \overline{\mathbb{C}P^2}$, with a canonical symplectic form ω_1 . The image of $\partial B(a_i)$ in this manifold is called the i^{th} exceptional divisor. If E_i denotes the homology class of the i^{th} exceptional divisor, then the cohomology class of ω_1 is given by

$$\text{PD}[\omega_1] = L - \sum_{i=1}^m a_i E_i.$$

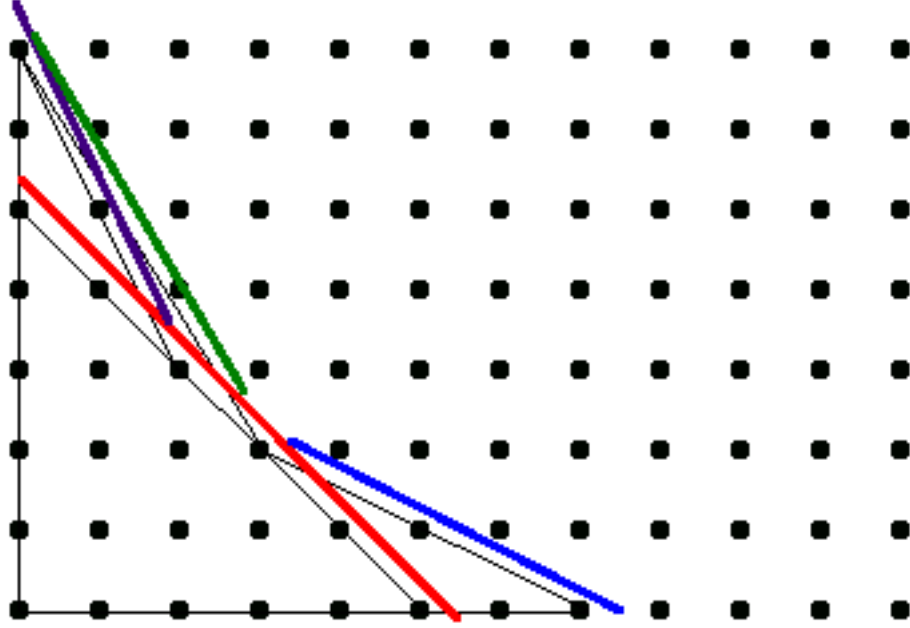


Figure 2.1: Blowing up a rational concave domain. In this case, we first blow up the region beneath the red line. We then blow up the region between the red line, the purple line, and the y -axis, and the region between the red line, the blue line, and the x -axis. The order in which we do these two blowups is irrelevant. Finally, we blow up the region between the green line, the purple line, and the red line. The result is a chain of four spheres. Note that the remaining portions of the purple and red lines correspond to spheres that are small, while the green and blue lines correspond to spheres that are large. The black lines give the canonical weight sequence decomposition of the domain.

Now let Ω be any rational concave toric domain, and include Ω into some large ball $\text{int}(B(R))$, which we can include densely into a $(\mathbb{C}P^2, \omega)$. We now mimic the definition of the weight sequence to define a sequence of symplectic blowups of $(\mathbb{C}P^2, \omega)$ that will produce one of the relevant chains of spheres. The reader is urged to see Figure 2.1, which will help illustrate the idea. Let a be the smallest real number such that Ω contains the triangle with vertices $(0, 0)$, $(0, a)$ and $(a, 0)$, let $\delta > 0$ be a small real number, and consider the triangle $\Delta(a + \delta)$ with vertices $(0, 0)$, $(0, a + \delta)$ and $(a + \delta, 0)$. Thus, in Figure 2.1, the upper boundary of $\Delta(a + \delta)$ is the red line. Then there is a symplectic embedding $B(a + \delta) \rightarrow B(R)$. Blow up along $B(a + \delta)$. Now the upper boundary of $\Delta(a + \delta)$ intersects the complement of Ω in the plane along a line segment between $(x_1, a + \delta - x_1)$ and $(x_2, a + \delta - x_2)$ with

$x_1 < x_2$. Let Γ_1 be the closure of the subset of Ω which is to the left of x_1 and above the line $x+y = a+\delta$, and let Γ_2 be the closure of the subset of Ω which is to the right of x_2 and above this line. Then, as in the definition of the weight sequence, Γ_1 and Γ_2 are affine equivalent to concave toric domains. In the present context, this implies that we can iterate the procedure from the above paragraph to perform a symplectic blowup for each element of the weight sequence for Ω . Each blowup produces a symplectic sphere. In choosing the relevant δ for each blowup, choose δ small enough so that none of the previous symplectic spheres are completely removed (so for example in Figure 2.1, we would want to choose the δ for the green sphere to be small enough so that it does not completely remove the red sphere). The result of this sequence of blowups is a symplectic manifold $(\mathbb{C}P^2 \# m \overline{\mathbb{C}P^2}, \omega_1)$ with a configuration of symplectic spheres $\mathcal{C}_{\Omega, \delta_\Omega}$, with one sphere for each element of the weight sequence. Here, δ_Ω denotes a sequence of small real numbers corresponding to the δ for each blow up.

We will want to define a similar sequence of blowups if Ω is a rational convex domain. Specifically, let b be the head of the weight sequence for Ω , and choose a small $\delta > 0$. The line $x+y = b-\delta$ intersects Ω in a line segment from $(x_1, b-\delta-x_1)$ to $(x_2, b-\delta-x_2)$, where $x_1 < x_2$. Let $\Delta(b-\delta)$ be the triangle with vertices $(0,0)$, $(b-\delta,0)$ and $(0, b-\delta)$. Let Γ_1 be the closure of the region of the complement of Ω in $\Delta(b-\delta)$ that is to the left of x_1 , and let Γ_2 be the closure of the region of the complement that is below $b-\delta-x_2$. We showed in the definition of the weight sequence that Γ_1 and Γ_2 are affine equivalent to concave toric domains. As above, this then means that we can copy the argument from the previous paragraph to associate a symplectic blow up to each term in the negative weight sequence for Ω , of the $\mathbb{C}P^2$ that $\text{int}(B(b-\delta))$ includes densely into. As in the previous paragraph, this requires a choice of small real numbers corresponding to the δ in this blow up construction. We again denote this set of small numbers by δ . The result of these additional blowups is a symplectic manifold $(\mathbb{C}P^2 \# n \overline{\mathbb{C}P^2}, \omega_2)$ with a configuration of symplectic spheres which we denote by $\hat{\mathcal{C}}_{\Omega, \delta_\Omega}$.

Our blowup procedure is closely related to the *inner and outer approximations* from [12]. To elaborate, consider first the blow up procedure for rational concave Ω . Our blowup procedure shows that we can define another concave toric domain, called an *outer approximation* to Ω , such that the sequence of blowups removes the interior of the outer approximation and collapses the boundary of the outer approximation to the configuration of spheres $\mathcal{C}_{\Omega, \delta}$. Denote the outer approximation to Ω by $\Omega_\delta^{\text{out}}$. For example, in the situation illustrated in Figure 2.1, the upper boundary of $\Omega_\delta^{\text{out}}$ is given by starting where the purple line hits the vertical axis, and then traversing the part of the purple line to the left of the green line, then the green line, then the part of the red line between the green and blue lines, and then

the blue line. Similarly, our blowup procedure shows that we can define another convex toric domain, called an *inner approximation* to Ω , such that the sequence of blowups removes the complement of the inner approximation in $B(b - \delta)$ and collapses the boundary of the inner approximation to the configuration of spheres $\widehat{\mathcal{C}}_{\Omega, \delta_\Omega}$.

Here is the significance of these chains of spheres to our embedding problem:

Proposition 2.1. *Let Ω_1 be a rational concave toric domain, and let Ω_2 be a rational convex toric domain. Let m be the length of the weight expansion for Ω_1 , and let n be the length of the negative weight expansion for Ω_2 . If there is a symplectic form ω on $\mathbb{C}P^2 \# (m + n) \overline{\mathbb{C}P^2}$ such that there is a symplectic embedding*

$$\mathcal{C}_{\Omega_1, \delta_{\Omega_1}} \sqcup \widehat{\mathcal{C}}_{\Omega_2, \delta_{\Omega_2}} \rightarrow (\mathbb{C}P^2 \# (m + n) \overline{\mathbb{C}P^2}, \omega),$$

then there is a symplectic embedding

$$X_{\Omega_1} \rightarrow \text{int}(X_{\Omega_2}).$$

Proof. By assumption, there is a symplectic embedding

$$\mathcal{C}_{\Omega_1, \delta_{\Omega_1}} \rightarrow (\mathbb{C}P^2 \# (m + n) \overline{\mathbb{C}P^2}, \omega).$$

As explained in [12, Lem. 2.2], we can make a small perturbation to this embedding so that these symplectic spheres intersect symplectically orthogonally. A version of the symplectic neighborhood theorem, see for example [21, Prop. 3.5], now implies that a neighborhood of these spheres can be identified with a neighborhood of the spheres in the manifold $(\mathbb{C}P^2 \# m \overline{\mathbb{C}P^2}, \omega_1)$ that was constructed above by blowing up the outer approximation. We can therefore remove the $\mathcal{C}_{\Omega_1, \delta_{\Omega_1}}$ and glue in a copy of $X_{\Omega_1, \delta_1}^{\text{out}}$ to get a new symplectic manifold \tilde{Z} which admits a symplectic embedding of X_{Ω_1} . (This is a special case of the “blow down” procedure explained in [21], see especially [21, Cor. 3.6].)

The construction from the previous paragraph can be done in the complement of $\mathcal{C}_{\Omega_2, \delta_{\Omega_2}}$. Moreover, we can repeat the argument from the previous paragraph to conclude that a neighborhood of $\mathcal{C}_{\Omega_2, \delta_{\Omega_2}}$ in \tilde{Z} is standard. Let Z denote the complement of $\mathcal{C}_{\Omega_2, \delta_{\Omega_2}}$ in \tilde{Z} . We know from [17, Thm. 9.4.2] that there is a unique symplectic form that is standard near the boundary on any star-shaped subset of \mathbb{R}^4 . It then follows that we can identify $X_{\text{int}(\Omega_2^{\text{in}}, \delta_{\Omega_2})}$ with Z . Since Ω_2 contains the inner approximation, the proposition now follows. □

2.2 Connectivity

We can now give a quick proof of Proposition 1.5, which states that the space of embeddings from a concave domain into a convex one is connected. We also prove Corollary 1.6.

Proof of Proposition 1.5. Proposition 1.5 follows from the proof of [12, Cor. 1.6]. While the proof of [12, Cor. 1.6] is for ellipsoids, the discussion in §2.1 shows that the proof generalizes to our case without any modifications.

For completeness, we sketch the argument. First, assume that Ω_1 and Ω_2 are rational, and let g_0 and g_1 be symplectic embeddings of X_{Ω_1} into $\text{int}(X_{\Omega_2})$. By applying Alexander's trick, see e.g. the proof of [20, Prop. A.1], we can assume that g_0 and g_1 agree with the inclusion of $X_{r\Omega_1}$ into $\text{int}(X_{\Omega_2})$ for sufficiently small r . Then, as in §2.1, we can blow up along $X_{r\Omega_1}$ and X_{Ω_2} to get a symplectic manifold $X_0 = (\mathbb{C}P^2 \# (m+n)\overline{\mathbb{C}P^2}, \omega)$ with two chains of exceptional spheres $\mathcal{C}_{r\cdot\Omega_1, \delta_{r\cdot\Omega_1}} \sqcup \widehat{\mathcal{C}}_{\Omega_2, \delta_{\Omega_2}}$. We can also blow up along g_0 and g_1 to get two different symplectic forms ω_1 and ω_2 on X_0 (here, we are implicitly identifying the underlying spaces of these blow ups with X_0 as in Step 2 of [14, §3]). In the present situation, the argument from [14, §3] shows that ω_1 and ω_2 are deformation equivalent (the deformation is essentially given by blowing up along $X_{t\Omega_1}$ and X_{Ω_2} as t varies). By using the singular inflation procedure from [13], we can convert this deformation to an isotopy, and we can assume that this isotopy is supported away from $\mathcal{C}_{r\cdot\Omega_1, \delta_{r\cdot\Omega_1}} \sqcup \widehat{\mathcal{C}}_{\Omega_2, \delta_{\Omega_2}}$, see [12, Cor. 1.6] and [16, Thm. 1.2.11]. We can therefore blow down this isotopy to give the desired isotopy between g_0 and g_1 . The result for nonrational Ω_1 and Ω_2 follows by approximating by rational domains. \square

Proof of Corollary 1.6. This also follows without any modifications from the proof of [12, Cor 1.6]: from the sequence of embeddings

$$X_{\lambda\Omega_1} \rightarrow \text{int}(X_{\Omega_2}),$$

we can obtain a sequence of embeddings $g_n : X_{(1-1/n)\Omega_1} \rightarrow \text{int}(X_{\Omega_2})$. By applying Proposition 1.5, we can assume that this sequence of maps is nested. We can therefore construct the desired symplectic embedding by taking the direct limit. \square

2.3 Inflating the spheres

We can now complete the proof of Theorem 1.4 by using the inflation procedure.

Proof of Theorem 1.4. Before beginning the proof, we briefly comment on one point, in order to motivate what follows.

Let Ω_1 be a rational concave domain and let Ω_2 be a rationally convex domain. By assumption, there is a ball packing of a ball, determined by the weights of the Ω_i . For the inflation method, we will want to blow up along this ball packing, to conclude that a certain cohomology class is represented by a symplectic form. However, we are given a packing by open balls, while to blow up we would like a packing by closed balls. To remedy this, observe that if $\varepsilon > 0$ is sufficiently small, we have a ball packing

$$\cup_i B((1-\varepsilon)a_i) \cup_j B((1-\varepsilon)b_j) \rightarrow \text{int}(B(b)), \quad (2.1)$$

where the a_i are the weights of Ω_1 , and the b, b_j are the weights of Ω_2 . The numbers $(1-\varepsilon)a_i$ are the weights of $(1-\varepsilon)\Omega_1$. Meanwhile, the numbers $(b, (1-\varepsilon)b_1, \dots, (1-\varepsilon)b_n)$ are the weights of a convex toric domain $\tilde{\Omega}_2$ with the property that $\tilde{\Omega}_2 \subset (1+\tilde{\varepsilon})\Omega_2$. Here, $\tilde{\varepsilon}$ can be made arbitrarily small if ε is made small enough. We will show below that we can construct a symplectic embedding $(1-\varepsilon)X_{\Omega_1} \rightarrow \text{int}(X_{\tilde{\Omega}_2}) \subset (1+\tilde{\varepsilon})X_{\Omega_2}$. We can then construct the desired symplectic embedding by appealing to Corollary 1.6. The details are as follows:

Step 1. Let r be a small enough rational number that $r \cdot (1-\varepsilon)\Omega_1 \subset \text{int}(\tilde{\Omega}_2)$. Then $r \cdot (1-\varepsilon)\Omega_1$ is a concave toric domain, and $\tilde{\Omega}_2$ is a convex toric domain. We can therefore apply the iterated blowup procedure from §2.1 to conclude that there is a symplectic embedding

$$\mathcal{C}_{r \cdot (1-\varepsilon)\Omega_1, \delta_{r \cdot (1-\varepsilon)\Omega_1}} \sqcup \hat{\mathcal{C}}_{\tilde{\Omega}_2, \delta_{\tilde{\Omega}_2}} \rightarrow (\mathbb{C}P^2 \# (m+n) \overline{\mathbb{C}P^2}, \omega_1).$$

Let L denote the homology class of the line in this blowup, let E_1, \dots, E_m be the exceptional classes associated to the blow ups for $r \cdot (1-\varepsilon)\Omega_1$, and let $\hat{E}_1, \dots, \hat{E}_n$ be the exceptional classes associated to the blow ups for $\tilde{\Omega}_2$. Let $\ell = \text{PD}(L)$, let $e_i = \text{PD}(E_i)$, and let $\hat{e}_j = \text{PD}(\hat{E}_j)$. By §2.1, we know that the cohomology class of ω_1 is given by

$$[\omega_1] = (b - \text{err}_2(\delta))\ell - \sum_{i=1}^m (r(1-\varepsilon)a_i + \text{err}_i(\delta_1))e_i - \sum_{j=1}^n ((1-\varepsilon)b_j + \text{err}_j(\delta_2))\hat{e}_j, \quad (2.2)$$

where the $(1-\varepsilon)a_i$ are the terms in the weight sequence for Ω_1 , the $(1-\varepsilon)b_j$ are the terms in the weight sequence for $\tilde{\Omega}_2$, and the err_i denote (possibly negative) error terms that are small and determined by the relevant δ_j . Meanwhile, the homology class of the image of each exceptional sphere in this manifold is determined by the canonical decompositions into affine triangles of $(1-\varepsilon)\Omega_1$ and $B(b) \setminus \tilde{\Omega}_2$ given by the weight sequences. In particular, the homology classes of these exceptional spheres do not depend on r .

Step 2. We now want to show that there is a symplectic embedding of $\mathcal{C}_{(1-\varepsilon)\Omega_1, \delta_{(1-\varepsilon)\Omega_1}} \sqcup \widehat{\mathcal{C}}_{\widetilde{\Omega}_2, \delta_{\widetilde{\Omega}_2}}$ into $(\mathbb{C}P^2 \# (m+n)\overline{\mathbb{C}P^2}, \omega)$ for some ω , so that we can appeal to Proposition 2.1. Since the intersection properties of the configuration of spheres $\mathcal{C}_{r \cdot (1-\varepsilon)\Omega_1, \delta_{r \cdot (1-\varepsilon)\Omega_1}} \sqcup \widehat{\mathcal{C}}_{\widetilde{\Omega}_2, \delta_{\widetilde{\Omega}_2}}$ do not depend on r , we therefore just have to alter the symplectic form so that these spheres have the correct area while remaining symplectic. To do this, we inflate the spheres, as in [16].

To perform the inflation, we need to find an appropriate J -holomorphic curve to inflate along. We now explain how to find such a curve. By assumption, as mentioned at the beginning of this proof, there is a ball packing (2.1). This gives a ball packing

$$\cup_i B((1-\varepsilon)(1+\varepsilon')a_i) \cup_j B((1-\varepsilon)b_j) \rightarrow \text{int}(B(b)), \quad (2.3)$$

where ε' is sufficiently small. Blowing up along this ball-packing shows that the class $a = b\ell - \sum_{i=1}^n (1-\varepsilon)(1+\varepsilon')a_i e_i - \sum_{j=1}^m (1-\varepsilon)b_j \widehat{e}_j$ is represented by a symplectic form. We can assume that this class is rational. As explained in the proof of [12, Prop. 1.10], work of Kronheimer and Mrowka [11] then shows that for all sufficiently large integers q , the class qa has nontrivial Seiberg-Witten invariant. Since qa is also represented by a symplectic form, Taubes' "Gromov = Seiberg-Witten" theorem then implies that $\text{PD}(qa)$ has nontrivial Gromov invariant.

Step 3. We would now like to conclude that the homology class $\text{PD}(qa)$ is represented by a connected embedded J -holomorphic curve, so that we can apply the "standard" inflation procedure, e.g. as explained in [14, Lem 1.1]. However, as explained in [13], there is a substantial technical hurdle to concluding this. We can circumvent this difficulty by using the "singular" inflation procedure from [16].

To elaborate, the difficulty is that the inflation procedure requires choosing an ω_1 tame almost complex structure J such that $\mathcal{C}_{r \cdot (1-\varepsilon)\Omega_1, \delta_{r \cdot (1-\varepsilon)\Omega_1}} \sqcup \widehat{\mathcal{C}}_{\widetilde{\Omega}_2, \delta_{\widetilde{\Omega}_2}}$ is J -holomorphic, and this can not be done while keeping J suitably generic so that Taubes' Gromov invariant for this J is defined. However, in the present context we can still apply [16, Lem. 1.2.11] to find the desired family of symplectic forms.

While for the applications in this paper, we just need to verify that the assumptions of [16, Lem. 1.2.11] hold, for completeness we sketch how the singular inflation procedure from [16] works in this situation. The basic point is that we can find a family of suitably generic J_t tending to a J such that the configuration of spheres is J -holomorphic. By Gromov compactness, we can therefore find a J -holomorphic nodal representative of the class $\text{PD}(qa)$. By perturbing J and this curve as in [16, §3], see especially [16, Prop. 3.1.3], we can assume that each component of this curve is a multiple cover of an

embedded curve. The hypotheses of [16, Lem. 1.2.11] will then ensure that each of these components has nonnegative intersection with $\text{PD}(qa)$, which will allow us to inflate. For the details of the inflation process, see [16, §5.2], especially [16, Prop. 5.1.2].

We now verify that the hypotheses of [16, Lem. 1.2.11] hold. This requires checking that the class $\text{PD}(qa)$ satisfies the four requirements of [16, Def. 1.2.4]. The only points that require further explanation are the third and fourth bullet points. The third bullet point requires that A has nonnegative intersection with any exceptional sphere. This holds due to (2.1), by a standard argument, see for example [10, Prop. 6]. The fourth bullet point requires checking that A has nonnegative intersection with each of the spheres in $\mathcal{C}_{r \cdot (1-\varepsilon)\Omega_1, \delta_{r \cdot (1-\varepsilon)\Omega_1}} \sqcup \widehat{\mathcal{C}}_{\widetilde{\Omega}_2, \delta_{\widetilde{\Omega}_2}}$. To see that this holds, remember that the homology classes of these spheres depend neither on r nor on δ . The claim now follows by (2.2), since each of these spheres have positive area with respect to the form ω_1 from (2.2), and this remains true as we let all the δ_i tend to 0 (note that as δ_i tends to 0, $\text{err}_i(\delta_i)$ does as well).

Step 4. We can therefore apply [16, Lem. 1.2.11] to inflate. In the present context, this procedure produces for all positive t a family of symplectic forms ω_t such that each ω_t restricts to a symplectic form along $\mathcal{C}_{r(1-\varepsilon)\Omega_1, \delta_{r(1-\varepsilon)\Omega_1}} \cup \widehat{\mathcal{C}}_{\widetilde{\Omega}_2, \delta_{\widetilde{\Omega}_2}}$ and the ω_t have cohomology class

$$[\omega_t] = [\omega_1] + tqa.$$

Now consider $\omega_t/(1+tq)$. We have

$$\begin{aligned} [\omega_t]/(1+tq) &= b\ell - \sum_{j=1}^n (1-\varepsilon)b_j \widehat{e}_j - \frac{r+(1+\varepsilon')tq}{1+tq} \sum_{i=1}^m (1-\varepsilon)a_i e_i - \\ &\quad \frac{\text{err}(\delta)}{1+tq} \ell - \sum_{i=1}^m \frac{\text{err}_i(\delta_i)}{1+tq} e_i - \sum_{j=1}^n \frac{\text{err}_j(\delta_j)}{1+tq} \widehat{e}_j. \end{aligned}$$

Now if t is sufficiently large, then $\frac{r+(1+\varepsilon')tq}{1+tq} = 1$. By choosing the δ_i for $\mathcal{C}_{(1-\varepsilon)\Omega_1, \delta_{(1-\varepsilon)\Omega_1}} \sqcup \widehat{\mathcal{C}}_{\widetilde{\Omega}_2, \delta_{\widetilde{\Omega}_2}}$ sufficiently small for this large t , it now follows that there is a symplectic embedding $\mathcal{C}_{(1-\varepsilon)\Omega_1, \delta_{(1-\varepsilon)\Omega_1}} \sqcup \widehat{\mathcal{C}}_{\widetilde{\Omega}_2, \delta_{\widetilde{\Omega}_2}} \rightarrow (\mathbb{C}P^2 \# (m+n)\overline{\mathbb{C}P^2}, \omega_t)$. By Proposition 2.1, there is now for all sufficiently small ε and ε' a symplectic embedding $X_{(1-\varepsilon)\Omega_1} \rightarrow X_{\widetilde{\Omega}_2} \subset (1+\varepsilon')\text{int}(X_{\Omega_2})$. Theorem 1.4 now follows by Corollary 1.6. \square

2.4 Examples

We now present several illustrative examples.

Example 2.2. Let Ω be the rectangle with vertices $(0, 0), (1, 0), (0, 1)$ and $(1, 1)$, and let Ω' be the triangle with vertices $(0, 0), (2, 0)$ and $(0, 1)$. Then X_Ω is a polydisk and $X_{\Omega'}$ is an ellipsoid. Both Ω and Ω' are convex (we could also regard Ω' as concave, although for this example we do not want to), and the weight sequence for both is given by $(2, 1, 1)$; in particular, both have the same weight sequence. This shows that weight sequences are not unique. Also, by Theorem 1.4, a concave domain embeds into X_Ω if and only if it embeds into $X_{\Omega'}$. This generalizes a result of Frenkel and Mueller [6, Cor. 1.5], which proves this when the domain is an ellipsoid (our proof is also different from theirs).

Example 2.3. Let (a_0, \dots, a_n) be any finite sequence of nonincreasing real numbers. We now explain why we can always construct a concave toric domain with weight sequence (a_0, \dots, a_n) . This concave domain will have the property that at each step in the inductive definition of the weight sequence, the domain Ω'_2 from §1.3 is empty (we will call such a domain *short*). By induction, we can assume that we can construct a short rational concave domain Ω_0 with weight sequence (a_1, \dots, a_n) . Now, consider the triangle $\Delta(a_0)$ with vertices $(0, 0), (a_0, 0)$ and $(0, a_0)$. Multiply Ω_0 by the matrix $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ and then translate the result by $(a_0, 0)$. Let Ω be formed by taking the union of this region with $\Delta(a_0)$. Then by construction Ω is a short concave domain with weight sequence (a_0, \dots, a_n) . Thus, any possible ball packing problem of a ball can arise by applying Theorem 1.4. This is to be compared with the case of embedding an ellipsoid into a ball, where the ball packings that arise are much more constrained, see [18].

Example 2.4. We now work through a more extended example in detail, see Figure 2.2.

Let Ω_1 be the domain whose upper boundary has vertices $(0, 10/3), (2/3, 4/3), (4/3, 2/3)$, and $(7/3, 0)$, and let Ω_2 be the domain whose upper boundary has vertices $(0, 1), (1, 2)$ and $(5, 0)$. Then the weight expansion of Ω_1 is $(2, 2/3, 2/3, 1/3, 1/3)$ and the weight expansion of Ω_2 is $(5, 3, 2, 1)$, see Figure 2.2.

By Theorem 1.4, to see if $\text{int}(X_{\Omega_1})$ embeds into $\text{int}(X_{\Omega_2})$, it is equivalent to see if there is a ball packing

$$\text{int}(B(2/3) \sqcup B(2/3) \sqcup B(2) \sqcup B(1/3) \sqcup \sqcup B(1/3) \sqcup B(3) \sqcup B(2) \sqcup B(1)) \rightarrow B(5). \quad (2.4)$$

One can check, e.g. by applying the algorithm from [2, §2.3], that in fact such a ball packing exists. Hence, there is a symplectic embedding $\text{int}(X_{\Omega_1}) \rightarrow \text{int}(X_{\Omega_2})$. In fact, this embedding is optimal (e.g. by [2, §2.3] again applied to Equation 2.4), in the sense that no larger scaling of $\text{int}(X_{\Omega_1})$ embeds into $\text{int}(X_{\Omega_2})$.

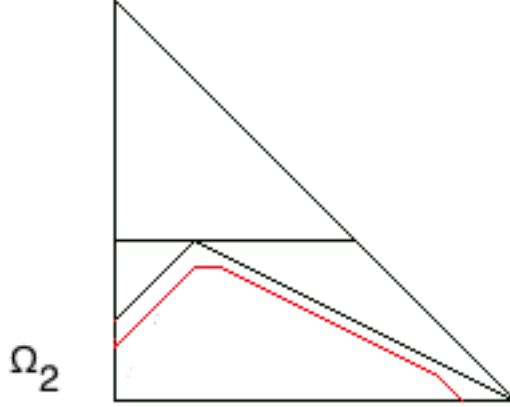


Figure 2.2: The target for Example 2.4. We have drawn the canonical decomposition given by the weight sequence (remember that the weight sequence for Ω_2 gives a decomposition of the complement of Ω_2 in a ball). The red line is the upper boundary of the inner approximation of Ω_2 .

To illustrate the concepts from the previous sections, note that there are five spheres in the chain of spheres corresponding to the blow up of $r \cdot \Omega_1$. Each sphere corresponds to a blow up, and if we label these spheres in the order that they appear as edges of the outer approximation (with the first sphere the left most edge), and label the blow ups they correspond to accordingly, then the spheres (from left to right) have homology classes $E_1, E_2 - E_1, E_3 - E_2 - E_4 - E_5, E_4$ and $E_5 - E_4$.

There are four spheres in the chain of spheres corresponding to the blow up of Ω_2 (including the sphere corresponding to the line at infinity). If we label these spheres and the blowups with the same ordering convention as above, then they have homology classes $\hat{E}_1, \hat{E}_2 - \hat{E}_1 - \hat{E}_3, \hat{E}_3$, and $L - \hat{E}_2 - \hat{E}_3$.

The cohomology class of the symplectic form on the blow up is given in this notation by

$$\begin{aligned}
[\omega_1] = & 5L - (2/3)re_1 - (2/3)re_2 - 2re_3 \\
& - (1/3)re_4 - (1/3)re_5 \\
& - \hat{e}_1 - 3\hat{e}_2 - 2\hat{e}_3 \\
& - \sum_{i=1}^5 \text{err}_i(\delta_1)e_i - \sum_{j=1}^3 \text{err}_j(\delta_2)\hat{e}_j.
\end{aligned} \tag{2.5}$$

3 Sharpness for the ball packing problem implies ECH capacities are sharp

We now explain the proof of Theorem 1.2. The key point is that it was shown in [8] that ECH capacities are known to be sharp for ball packing problems.

Proof. We need to show that $\text{int}(X_{\Omega_1})$ embeds into $\text{int}(X_{\Omega_2})$ if and only if $c_k(\text{int}(X_{\Omega_1})) \leq c_k(\text{int}(X_{\Omega_2}))$ for all k . The fact that a symplectic embedding

$$\text{int}(X_{\Omega_1}) \rightarrow \text{int}(X_{\Omega_2})$$

implies that $c_k(\text{int}(X_{\Omega_1})) \leq c_k(\text{int}(X_{\Omega_2}))$ for all k follows from the Monotonicity property of ECH capacities shown in [8].

For the converse, first note that by Corollary 1.6, we can assume that Ω_1 and Ω_2 are rational. Now by the Monotonicity and Disjoint Union properties from [8], and the argument for the “only if” direction of Theorem 1.4, we know that

$$c_{ECH}(\text{int}(X_{\Omega_2})) \# c_{ECH}(\text{int}(\widehat{B}(\Omega_2))) \leq c_{ECH}(B(b)),$$

where $\#$ denotes the “sequence sum” defined in [8], and c_{ECH} denotes the sequence of ECH capacities. We also know by the same argument that

$$c_k(\text{int}(B(\Omega_1))) \leq c_k(\text{int}(X_{\Omega_1})).$$

We also know that sequence sum against a fixed sequence respects inequalities. Hence, combining $c_k(\text{int}(X_{\Omega_1})) \leq c_k(\text{int}(X_{\Omega_2}))$ with the above equations and the Disjoint Union property implies that

$$c_k(\text{int}(B(\Omega_1)) \sqcup \text{int}(\widehat{B}(\Omega_2))) \leq c_k(B(b)). \quad (3.1)$$

It is known that ECH capacities give sharp obstructions to all (open) ball packings of a ball, see e.g. [8]. Hence, (3.1) implies that there exists a symplectic embedding

$$\text{int}(B(\Omega_1)) \sqcup \text{int}(\widehat{B}(\Omega_2)) \rightarrow B(b).$$

Hence by Theorem 1.4, there exists a symplectic embedding

$$\text{int}(X_{\Omega_1}) \rightarrow \text{int}(X_{\Omega_2}),$$

hence the theorem. □

A Appendix (by Keon Choi and Daniel Cristofaro-Gardiner): The geometric meaning of ECH capacities of convex domains

We assume below that the reader is familiar with the definitions and notation from the body of this paper. There, the second author showed that ECH capacities give a sharp obstruction to embedding any concave toric domain into a convex one. As explained in §3, to show this, all we need to know about ECH capacities are the basic axioms they satisfy, together with the fact that they are sharp for ball packings of a ball (of course, we also need to know Theorem 1.4, which states that embedding a concave domain into a convex one is equivalent to a ball packing problem). This suggests that there should be a close relationship between the ECH capacities of concave or convex toric domains, and the ECH capacities of balls.

In [3], the authors and Frenkel, Hutchings and Ramos showed that ECH capacities of any concave toric domain are given by the ECH capacities of the disjoint union of the balls determined by the weight sequence of the domain. We now prove a similar formula for convex toric domains.

To state the formula, recall the “sequence subtraction” operation from [8]. This is given for sequences S and T by

$$(S - T)_k = \min_{l \geq 0} S_{k+l} - T_l.$$

In the present context, the sequence subtraction operation is significant because of the following:

Theorem A.1. *Let Ω be a convex toric domain, let b be the head of the weight expansion for Ω , and let b_i be the i^{th} term in the negative weight expansion for X_Ω . Then*

$$c_{ECH}(X_\Omega) = c_{ECH}(B(b)) - c_{ECH}(\sqcup_i B(b_i)). \quad (\text{A.1})$$

Note that it follows from the Monotonicity and Scaling axioms that $c_k(X_\Omega) = c_k(\text{int}(X_\Omega))$ for any convex toric domain X_Ω . Note also that even when Ω is not rational, the above formula still makes sense, see [3, Rmk. 1.6].

We can regard Theorem A.1 as expressing a fundamental limitation of the strength of ECH capacities of convex domains: all the ECH capacities of a convex domain Ω can tell us about embeddings into X_Ω is whether or not ECH obstructs the corresponding ball packing problem. This theorem is similar to [3, Thm. 1.4] but we give a significantly shorter indirect argument quoting the results in [3] and [8]. Before proceeding with the proof, we recall relevant definitions.

We begin by defining the upper boundaries of the regions we need to consider to compute ECH capacities of concave and convex domains:

Definition A.2. Let $\Lambda : [0, c] \rightarrow \mathbb{R}^2$ be a piecewise linear path for some $c \geq 0$, parametrized by its Euclidean length. If Λ consists of n line segments, its tangent Λ' is a locally constant function defined on $[0, c] \setminus \{0 = c_0 < \dots < c_n = c\}$. Also, for any nonzero vector $v \in \mathbb{R}^2$, let $\theta(v)$ be the number $\theta \in [0, 2\pi)$ so that v is a positive multiple of $(\sin \theta, \cos \theta)$.

- For each $0 \leq i \leq n$, $\Lambda(c_i)$ is called a *vertex* of Λ . An *edge* of Λ is the line segment ν between vertices of Λ and $\vec{\nu}$ denotes the displacement vector of ν .
- Λ is a *lattice path* if its vertices are lattice points and $\Lambda(0) = (0, y(\Lambda))$ and $\Lambda(c) = (x(\Lambda), 0)$ for nonnegative integers $x(\Lambda)$ and $y(\Lambda)$.
- Λ is *concave* if $\theta(\Lambda')$ is nonincreasing and takes values in $(\pi/2, \pi)$.
- Λ is *convex* if $\theta(\Lambda')$ is nondecreasing and takes values in $(0, 3\pi/2)$.

The paths Λ have an Ω -length, defined by the domain Ω , which will also be important.

Definition A.3. Let X_Ω be a convex domain and Λ a convex lattice path. If ν is an edge of Λ , let $p_{\Omega, \nu}$ be a point on the boundary of Ω such that Ω lies entirely in the “right half-plane” of the line through $p_{\Omega, \nu}$ in the direction $\vec{\nu}$: more precisely, for any $p \in \Omega$, $(p - p_{\Omega, \nu}) \times \vec{\nu} \geq 0$ where \times denotes the cross product. Define

$$\ell_\Omega(\Lambda) = \sum_{\nu \in \text{Edges}(\Lambda)} \vec{\nu} \times p_{\Omega, \nu}. \quad (\text{A.2})$$

If X_Ω is a concave domain and Λ is a concave lattice path, $\ell_\Omega(\Lambda)$ is defined by (A.2), where $p_{\Omega, \nu}$ is a point on the boundary of $\Omega^c := [0, \infty)^2 \setminus \Omega$ so that Ω^c lies entirely on the “left half-plane” of the line through $p_{\Omega, \nu}$ in the direction $\vec{\nu}$.

We will also want to count lattice points in regions bounded by Λ . We now make this precise.

Definition A.4. If Λ is a convex lattice path, let $\check{\mathcal{L}}_\Omega(\Lambda)$ denote the count of lattice points in the region enclosed by Λ and the axes, including all the lattice points on the boundary. If Λ is a concave lattice path, let $\hat{\mathcal{L}}(\Lambda)$ denote the number of lattice points in the region enclosed by Λ and the axes, not including lattice points on Λ itself.

We can now give the proof of the main theorem of this appendix.

Proof of Theorem A.1. Recall from §1.3 that the first step of the weight expansion for X_Ω determines regions Ω_1, Ω_2 and Ω_3 such that X_{Ω_1} is a $B(b)$ and X_{Ω_2} and X_{Ω_3} are concave toric domains. For a given $k \geq 0$, we claim a series of inequalities

$$\begin{aligned}
c_k(X_\Omega) &\leq \min_{k_1-l=k} \{c_{k_1}(X_{\Omega_1}) - c_l(\sqcup B(b_i))\} \\
&= \min_{k_1-k_2-k_3=k} \{c_{k_1}(X_{\Omega_1}) - c_{k_2}(X_{\Omega_2}) - c_{k_3}(X_{\Omega_3})\} \\
&\leq \min\{\ell_\Omega(\Lambda) | \check{\mathcal{L}}(\Lambda) \geq k+1\} \\
&\leq c_k(X_\Omega),
\end{aligned} \tag{A.3}$$

which proves the theorem. Here and throughout the proof, k_1, k_2, k_3 and l denote nonnegative integers. We now explain the proofs of the above inequalities.

Step 1. By the definition of the weight expansion, there is a symplectic embedding

$$X_\Omega \sqcup_i \text{int}(B(b_i)) \rightarrow B(b).$$

It then follows from the Monotonicity axiom from [8] that for any k_1 and l

$$c_{k_1}(X_\Omega) + c_l(\sqcup_i B(b_i)) \leq c_{k_1+l}(B(b)).$$

This proves the first inequality of (A.3).

Step 2. Since the weights of Ω_2 and Ω_3 collectively correspond to the negative weights of Ω ,

$$\max_{k_2+k_3=l} \{c_{k_2}(X_{\Omega_2}) + c_{k_3}(X_{\Omega_3})\} = \max_{\sum l_i=l} \sum c_{l_i}(B(b_i)) = c_l(\sqcup B(b_i))$$

by [3, Thm. 1.4]. This proves the equality on the second line of (A.3).

Step 3. To prove the third inequality of (A.3), given any convex lattice path Λ with $\check{\mathcal{L}}(\Lambda) - 1 = k_0 \geq k$, we show how to choose k_1, k_2 and k_3 with $k_1 - k_2 - k_3 = k$ so that

$$\ell_\Omega(\Lambda) \geq c_{k_1}(X_{\Omega_1}) - c_{k_2}(X_{\Omega_2}) - c_{k_3}(X_{\Omega_3}). \tag{A.4}$$

Write Λ as a concatenation $\tilde{\Lambda}_2 \tilde{\Lambda}_1 \tilde{\Lambda}_3$ of paths so that $\theta(\tilde{\Lambda}_2'), \theta(\tilde{\Lambda}_1')$ and $\theta(\tilde{\Lambda}_3')$ take values in $(0, 3\pi/4)$, $\{3\pi/4\}$ and $(3\pi/4, 3\pi/2)$, respectively. As in the definition of the weight expansion, $\tilde{\Lambda}_2$ and $\tilde{\Lambda}_3$ are affine equivalent to concave lattice paths Λ_2 and Λ_3 , respectively. Also, let Λ_1 denote the linear path from $(0, a)$ to $(a, 0)$ extending $\tilde{\Lambda}_1$. We take $k_2 = \hat{\mathcal{L}}(\Lambda_2)$, $k_3 = \hat{\mathcal{L}}(\Lambda_3)$ and $k_1 = k + k_2 + k_3$. Observe that $\check{\mathcal{L}}(\Lambda_1) = k_0 + k_2 + k_3 \geq k_1$.

By the Ellipsoid axiom from [3] and the Monotonicity axiom,

$$\ell_{\Omega_1}(\Lambda_1) = c_{k_0+k_2+k_3}(B(b)) \geq c_{k_1}(B(b)).$$

It was also shown in [3, Thm. 1.21] that

$$\ell_{\Omega_2}(\Lambda_2) \leq c_{k_2}(X_{\Omega_2})$$

and

$$\ell_{\Omega_3}(\Lambda_3) \leq c_{k_3}(X_{\Omega_3}).$$

Moreover, by essentially the same argument as in Step 4 of [3, §2.1],

$$\ell_{\Omega}(\Lambda) = \ell_{\Omega_1}(\Lambda_1) - \ell_{\Omega_2}(\Lambda_2) - \ell_{\Omega_3}(\Lambda_3).$$

We substitute the previously obtained bounds into the above to obtain (A.4).

Step 4. Consider a dilation $\tilde{\Omega}$ of Ω by a factor $\lambda < 1$ about an interior point of Ω . Then, $X_{\tilde{\Omega}}$ is a disk bundle over T^2 , and by [8, Thm. 1.11], there is a closed convex path $\tilde{\Lambda}$ with corners on lattice points so that $\check{\mathcal{L}}(\tilde{\Lambda}) = k + 1$ and $\ell_{\tilde{\Omega}}(\tilde{\Lambda}) = c_k(\tilde{\Omega})$. Here, $\check{\mathcal{L}}(\tilde{\Lambda})$ denotes the number of lattice points in the region enclosed by $\tilde{\Lambda}$, including the ones on the boundary, and $\ell_{\tilde{\Omega}}(\tilde{\Lambda})$ is defined by (A.2) as in the case of a convex domain.

Consider the part Λ of the path $\tilde{\Lambda}$ consisting only of edges with $0 < \theta(\nu) < 3\pi/2$. Then Λ is a convex lattice path (after translation if necessary) with $\check{\mathcal{L}}(\Lambda) \geq k + 1$ and $\ell_{\tilde{\Omega}}(\tilde{\Lambda}) = \lambda \ell_{\Omega}(\Lambda)$. Hence, by the Monotonicity axiom,

$$\ell_{\Omega}(\Lambda) = c_k(\tilde{\Omega})/\lambda \leq c_k(\Omega)/\lambda$$

Taking the limit as $\lambda \rightarrow 1$ proves the last inequality. \square

We close with the following analogue of the formula from [8, Thm. 1.11].

Corollary A.5. *Let Ω be a convex toric domain. Then*

$$c_k(X_{\Omega}) = \min\{\ell_{\Omega}(\Lambda) \mid \check{\mathcal{L}}(\Lambda) = k + 1\},$$

where the minimum is over convex lattice paths Λ .

Proof. As part of the proof of Theorem A.1, we saw this formula where the minimum is taken over Λ with $\check{\mathcal{L}}(\Lambda) \geq k + 1$. We can replace the inequality with equality, since $c_k(X_{\Omega}) \leq c_{k'}(X_{\Omega})$ whenever $k \leq k'$. \square

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